

ANALYTIC FOURIER-FEYNMAN TRANSFORMS AND CONVOLUTION TYPE OPERATIONS ASSOCIATED WITH GAUSSIAN PROCESSES ON WIENER SPACE

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ABSTRACT. In this paper we introduce the concept of a convolution type operation of functionals on Wiener space. It contains several kinds of the concepts of convolution products on Wiener space, which have been studied by many authors. We then investigate fundamental relationships between generalized analytic Fourier-Feynman transforms and convolution type operations. Both of the generalized analytic Fourier-Feynman transform of the convolution type operation and the convolution type operation of the generalized analytic Fourier-Feynman transforms are represented as a product of the generalized analytic Fourier-Feynman transforms.

1. INTRODUCTION

For $f \in L_1(\mathbb{R}^n)$, let the Fourier transform $\mathcal{F}(f)$ of f be given by

$$\mathcal{F}(f)(\vec{u}) = \int_{\mathbb{R}^n} e^{i\vec{u} \cdot \vec{v}} f(\vec{v}) dm_L^n(\vec{v})$$

where $dm_L^n(\vec{v})$ denotes the normalized Lebesgue measure $(2\pi)^{-n/2} d\vec{v}$ on \mathbb{R}^n . Also, for $f, g \in L_1(\mathbb{R}^n)$, let the convolution $f * g$ of f and g be given by

$$(f * g)(\vec{u}) = \int_{\mathbb{R}^n} f(\vec{u} - \vec{v}) g(\vec{v}) dm_L^n(\vec{v}).$$

Then the Fourier transform \mathcal{F} acts like a group homomorphism with convolution $*$ and ordinary multiplication on $L_1(\mathbb{R}^n)$. More precisely, one can see that for $f, g \in L_1(\mathbb{R}^n)$

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g).$$

Let $C_0[0, T]$ denote one-parameter Wiener space; that is, the space of all real-valued continuous functions x on $[0, T]$ with $x(0) = 0$. Let \mathcal{M} denote the class of all Wiener measurable subsets of $C_0[0, T]$ and let m be the Wiener measure. Then $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space.

A subset B of $C_0[0, T]$ is said to be scale-invariant measurable (s.i.m.) (see [13]) provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to be hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

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The concept of the analytic Fourier–Feynman transform (FFT) on the Wiener space $C_0[0, T]$, initiated by Brue [2], has been developed in the literature. This transform and its properties are similar in many respects to the ordinary Fourier function transform. For an elementary introduction to the analytic FFT, see [22] and the references cited therein. First of all, we refer to [22] for the precise definitions and the notations of the analytic FFT and the convolution product (CP) on the Wiener space $C_0[0, T]$. In [9], Huffman, Park and Skoug defined a CP for functionals on $C_0[0, T]$, and they then obtained various results for the analytic FFT and the CP [10, 11, 12]. In previous researches involving [6, 9, 10, 11, 12], the authors have been established the relationship between the analytic FFT and the corresponding CP of functionals F and G on $C_0[0, T]$, in the form

$$(1.1) \quad T_q((F * G)_q)(y) = T_q(F)\left(\frac{y}{\sqrt{2}}\right) T_q(G)\left(\frac{y}{\sqrt{2}}\right)$$

for s-a.e. $y \in C_0[0, T]$.

An essential structure hidden in the proof of equation (1.1) is based on the fact that the Gaussian processes

$$\mathfrak{Z}_+ \equiv \left\{ \frac{x_1 + x_2}{\sqrt{2}} : x_1, x_2 \in C_0[0, T] \right\} \text{ and } \mathfrak{Z}_- \equiv \left\{ \frac{x_1 - x_2}{\sqrt{2}} : x_1, x_2 \in C_0[0, T] \right\}$$

are independent, and the processes \mathcal{Z}_+ and \mathcal{Z}_- are equivalent to the standard Wiener process. More precisely, the product Wiener measure $m \times m$ is rotation invariant in $C_0^2[0, T]$, see [4, Lemmas 1 and 2]. As discussed in [4], those rotation invariant properties of $m \times m$ were concretely realized by Bearman [1].

Recently in [7], the authors used other rotation form of Wiener measure m to defined a multiple *analytic FFT associated with nonstationary Gaussian processes* \mathcal{Z}_h (\mathcal{Z}_h -FFT) on $C_0[0, T]$. The rotation form used in [7] is a generalization of Bearman's celebrated result and is intended to interpret behaviors of nonstationary Gaussian processes on $C_0[0, T]$. The authors also investigated various relationships which exist between the multiple FFT and the corresponding CP associated with nonstationary Gaussian processes on $C_0[0, T]$.

In this paper, motivated by the results in [6, 9, 10, 11, 12], we shall study the relationship between fundamental relationships between analytic \mathcal{Z}_h -FFTs and convolution type operations (CTO) on Wiener space. The paper is organized as follows. In Section 2 we briefly recall well-known results in Gaussian processes on Wiener space and give the concepts of the \mathcal{Z}_h -FFT and the convolution type operation of functionals on Wiener space. In Section 3, we emphasize the main purpose of this paper via specific examples. To do this we introduce the partially exponential type functionals on $C_0[0, T]$. In Section 4, as preliminary results, we investigate rotation properties of the Gaussian processes on Wiener space. In Section 5, we investigate fundamental relationships between analytic \mathcal{Z}_h -FFTs and CTOs. Finally in Section 6, we give several examples to apply our assertions in this paper.

2. PRELIMINARIES

For each $v \in L_2[0, T]$ and $x \in C_0[0, T]$, we let $\langle v, x \rangle = \int_0^T v(t) dx(t)$ denote the Paley–Wiener–Zygmund stochastic integral [15, 16, 17]. For any $h \in L_2[0, T]$ with $\|h\|_2 > 0$, let \mathcal{Z}_h be the stochastic process [8, 12, 18, 20] on $C_0[0, T] \times [0, T]$ given

by

$$(2.1) \quad \mathcal{Z}_h(x, t) = \int_0^t h(s) dx(s) = \langle h\chi_{[0,t]}, x \rangle.$$

Of course if $h(t) \equiv 1$ on $[0, T]$, then $\mathcal{Z}_h(x, t) = x(t)$ is an ordinary Wiener process. It is known that the PWZ stochastic integral $\langle v, x \rangle$ is Gaussian with mean zero and variance $\|v\|_2^2$, where $\|\cdot\|_2$ denotes the $L_2[0, T]$ -norm. Throughout this paper, we denote the Wiener integral of a Wiener measurable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_0[0, T]} F(x) dm(x).$$

Let $\beta_h(t) = \int_0^t h^2(u) du$. It is easy to see that \mathcal{Z}_h is a Gaussian process with mean zero and covariance function

$$E_x[\mathcal{Z}_h(x, s)\mathcal{Z}_h(x, t)] = \int_0^{\min\{s, t\}} h^2(u) du = \beta(\min\{s, t\}).$$

In addition, $\mathcal{Z}_h(\cdot, t)$ is stochastically continuous in t on $[0, T]$ and for any $h_1, h_2 \in L_2[0, T]$,

$$(2.2) \quad E_x[\mathcal{Z}_{h_1}(x, s)\mathcal{Z}_{h_2}(x, t)] = \int_0^{\min\{s, t\}} h_1(u)h_2(u) du.$$

Furthermore, if h is of bounded variation on $[0, T]$, then $\mathcal{Z}_h(x, t)$ is continuous in t for all $x \in C_0[0, T]$, i.e., $\mathcal{Z}_h(\cdot, \cdot)$ is a continuous process on $C_0[0, T] \times [0, T]$. Thus throughout the remainder of this paper we require h to be in $BV[0, T]$, the space of functions of bounded variation on $[0, T]$, for each process \mathcal{Z}_h .

It is known [8] that for $v \in L_2[0, T]$ and $h \in L_\infty[0, T]$,

$$(2.3) \quad \langle v, \mathcal{Z}_h(x, \cdot) \rangle = \langle vh, x \rangle$$

for s-a.e. $x \in C_0[0, T]$.

Using (2.3) and the change of variable formula, one can establish the integration formula on $C_0[0, T]$:

$$(2.4) \quad \int_{C_0[0, T]} e^{\langle v, \rho x \rangle} dm(x) = \exp \left\{ \frac{\rho^2}{2} \|v\|_2^2 \right\}$$

for every $v \in L_2[0, T]$ and $\rho \in \mathbb{R} \setminus \{0\}$.

Throughout this paper, we will assume that each functional F (or G) we consider satisfies the conditions:

$$(2.5) \quad F : C_0[0, T] \rightarrow \mathbb{C} \text{ is s.i.m. and s-a.e. defined,}$$

and for all $h \in L_2[0, T]$,

$$(2.6) \quad E_x[|F(\rho \mathcal{Z}_h(x, \cdot))|] < +\infty \text{ for each } \rho > 0.$$

Throughout this paper, let \mathbb{C} , \mathbb{C}_+ and $\tilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part, and nonzero complex numbers with nonnegative real part, respectively. For each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{1/2}$ denotes the principal square root of λ ; i.e., $\lambda^{1/2}$ is always chosen to have positive real part, so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in \mathbb{C}_+ for all $\lambda \in \tilde{\mathbb{C}}_+$.

Definition 2.1. Let F satisfy conditions (2.5) and (2.6) above. Let \mathcal{Z}_h be the Gaussian process given by (2.1) and for $\lambda > 0$, let $J(h; \lambda) = E_x[F(\lambda^{-1/2} \mathcal{Z}_h(x, \cdot))]$. If there exists a function $J^*(h; \lambda)$ analytic on \mathbb{C}_+ such that $J^*(h; \lambda) = J(h; \lambda)$ for all $\lambda > 0$, then $J^*(h; \lambda)$ is defined to be the analytic \mathcal{Z}_h -Wiener integral (namely, the generalized analytic Wiener integral with respect to the Gaussian paths $\mathcal{Z}_h(x, \cdot)$) of F over $C_0[0, T]$ with parameter λ . In this case, for $\lambda \in \mathbb{C}_+$ we write

$$E_x^{\text{an}w\lambda}[F(\mathcal{Z}_h(x, \cdot))] = J^*(h; \lambda).$$

Let $q \neq 0$ be a real number and let F be a functional such that the analytic \mathcal{Z}_h -Wiener integral $E_x^{\text{an}w\lambda}[F(\mathcal{Z}_h(x, \cdot))]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic \mathcal{Z}_h -Feynman integral (namely, the generalized analytic Feynman integral with respect to the Gaussian paths $\mathcal{Z}_h(x, \cdot)$) of F with parameter q and we write

$$(2.7) \quad E_x^{\text{anf}q}[F(\mathcal{Z}_h(x, \cdot))] = \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} E_x^{\text{an}w\lambda}[F(\mathcal{Z}_h(x, \cdot))].$$

Note that if $h \equiv 1$ on $[0, T]$, then these definitions agree with the previous definitions of the analytic Wiener integral and the analytic Feynman integral [3, 5, 9, 10, 11, 13, 14, 19, 21].

Next we state the definition of the generalized analytic FFT.

Definition 2.2. For $k \in L_2[0, T]$, $\lambda \in \mathbb{C}_+$, and $y \in C_0[0, T]$, let

$$(2.8) \quad T_{\lambda,k}(F)(y) = E_x^{\text{an}w\lambda}[F(y + \mathcal{Z}_k(x, \cdot))].$$

We define the L_1 analytic \mathcal{Z}_k -FFT (namely, the analytic FFT with respect to the Gaussian paths \mathcal{Z}_k), $T_{q,k}^{(1)}(F)$ of F , by the formula ($\lambda \in \mathbb{C}_+$)

$$T_{q,k}^{(1)}(F)(y) = \lim_{\lambda \rightarrow -iq} T_{\lambda,k}(F)(y),$$

for s-a.e. $y \in C_0[0, T]$ whenever this limit exists.

We note that if $T_{q,k}^{(1)}(F)$ exists and if $F \approx G$, then $T_{q,k}^{(1)}(G)$ exists and $T_{q,k}^{(1)}(G) \approx T_{q,k}^{(1)}(F)$. One can see that for each $k \in L_2[0, T]$, $T_{q,k}^{(1)}(F) \approx T_{q,-k}^{(1)}(F)$ since

$$\int_{C_0[0,T]} F(x) dm(x) = \int_{C_0[0,T]} F(-x) dm(x).$$

By Definitions 2.1 and 2.2, it is easy to see that for a nonzero real number q ,

$$(2.9) \quad T_{q,k}^{(1)}(F)(y) = E_h^{\text{anf}q}[F(y + \mathcal{Z}_k(x, \cdot))]$$

for s-a.e. $y \in C_0[0, T]$ if both sides exist.

Next we give the definition of the CTO.

Definition 2.3. Let F and G be functionals on $C_0[0, T]$. For $\lambda \in \tilde{\mathbb{C}}_+$, $g_1, g_2 \in BV[0, T]$ and $h_1, h_2 \in L_2[0, T]$, we define their CTO with respect to $\{\mathcal{Z}_{g_1}, \mathcal{Z}_{g_2}, \mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}\}$ (if it exists) by

$$(2.10) \quad \begin{aligned} & (F * G)_\lambda^{(g_1, g_2; h_1, h_2)}(y) \\ &= \begin{cases} E_x^{\text{an}w\lambda}[F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{h_1}(x, \cdot))G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{h_2}(x, \cdot))], & \lambda \in \mathbb{C}_+ \\ E_x^{\text{anf}q}[F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{h_1}(x, \cdot))G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{h_2}(x, \cdot))], & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases} \end{aligned}$$

When $\lambda = -iq$, we will denote $(F * G)_\lambda^{(g_1, g_2; h_1, h_2)}$ by $(F * G)_q^{(g_1, g_2; h_1, h_2)}$.

Remark 2.4. (i) Given a function h in $L_2[0, T] \setminus \{0\}$, letting $h_1 = -h_2 = h/\sqrt{2}$ and $g_1 = g_2 \equiv 1/\sqrt{2}$, equation (2.4) reduces the convolution product studied in [6, 7, 12, 20]:

$$(F * G)_q^{(g_1, g_2; h_1, h_2)}(y) = E_x^{\text{anf}_q} \left[F \left(\frac{y + \mathcal{Z}_k(x, \cdot)}{\sqrt{2}} \right) G \left(\frac{y - \mathcal{Z}_k(x, \cdot)}{\sqrt{2}} \right) \right].$$

(ii) Choosing $h_1 = -h_2 = g_1 = g_2 \equiv 1/\sqrt{2}$, we have the convolution product studied in [9, 10, 11]:

$$(F * G)_q^{(g_1, g_2; h_1, h_2)}(y) = E_x^{\text{anf}_q} \left[F \left(\frac{y + x}{\sqrt{2}} \right) G \left(\frac{y - x}{\sqrt{2}} \right) \right].$$

(iii) Choosing $h_1 = h_2 = g_1 = -g_2 = 1/\sqrt{2}$ and $\lambda = 1$, we have the convolution product studied in [23]:

$$(F * G)^{(g_1, g_2; h_1, h_2)}(y) = E_x \left[F \left(\frac{y + x}{\sqrt{2}} \right) G \left(\frac{-y + x}{\sqrt{2}} \right) \right].$$

3. OBSERVATION ON THE CLASS $\mathcal{E}(C_0[0, T])$

Let \mathcal{E} be the class of all functionals having the form

$$(3.1) \quad \Psi_u(x) = e^{\langle u, x \rangle} \quad \text{for a.e. } x \in C_0[0, T]$$

with $u \in L_2[0, T]$, and given $q \in \mathbb{R} \setminus \{0\}$, $v \in L_2[0, T]$, and $k \in BV[0, T]$, let $\mathcal{E}_{q, v, k}$ be the class of all functionals having the form

$$(3.2) \quad \Psi_u^{q, v, k}(x) = \Psi_u(x) \exp \left\{ \frac{i}{2q} \|vk\|_2^2 \right\} \quad \text{for a.e. } x \in C_0[0, T],$$

where Ψ_u is given by equation (3.1). We note that Ψ_u and $\Psi_u^{q, v, k}$ are scale invariant measurable.

The functionals given by equation (3.2) and linear combinations (with complex coefficients) of the $\Psi_u^{q, v, k}$'s are called the partially exponential type functionals on $C_0[0, T]$. The functionals given by (3.1) are also partially exponential type functionals because $\Psi_u^{q, v, 0}(x) = \Psi_u(x)$ for s-a.e. $x \in C_0[0, T]$.

For each $(q, v, k) \in (\mathbb{R} \setminus \{0\}) \times L_2[0, T] \times BV[0, T]$, the class $\mathcal{E}_{q, v, k}$ is dense in $L_2(C_0[0, T])$. Furthermore, $\text{Span} \mathcal{E}_{q, v, k}$, the linear manifold generated by $\mathcal{E}_{q, v, k}$ in $L_2(C_0[0, T])$, is closed under the ordinary multiplication because

$$\Psi_{u_1}^{q, v, k}(x) \Psi_{u_2}^{q, v, k}(x) = \alpha \exp \left\{ \langle u_1 + u_2, x \rangle + \frac{i}{2q} \|vk\|_2^2 \right\} = \alpha \Psi_{u_1 + u_2}^{q, v, k}(x)$$

for s-a.e. $y \in C_0[0, T]$, where with the complex coefficient α given by $\exp\{(i/2q)\|vk\|_2^2\}$. Thus the class $\text{Span} \mathcal{E}_{q, v, k}$ is a commutative algebra over the complex field \mathbb{C} .

In fact, using the fact that

$$\Psi_{u_1}^{q_1, v_1, k_1}(x) \Psi_{u_2}^{q_2, v_2, k_2}(x) = \beta \exp\{\langle u_1 + u_2, x \rangle\} = \beta \Psi_{u_1 + u_2}(x)$$

with

$$\beta = \exp \left\{ \frac{i}{2q_1} \|v_1 k_1\|_2^2 + \frac{i}{2q_2} \|v_2 k_2\|_2^2 \right\},$$

one can see that

$$\text{Span}\left(\bigcup_{\substack{q \in \mathbb{R} \\ v \in L_2[0, T] \\ k \in BV[0, T]}} \mathcal{E}_{q, v, k}\right) = \text{Span}\mathcal{E}.$$

We denote the set of all partially exponential type functionals on $C_0[0, T]$ by $\mathcal{E}(C_0[0, T])$, i.e. $\mathcal{E}(C_0[0, T]) = \text{Span}\mathcal{E}$.

First, using (2.7) with F replaced with Ψ_u , (2.3), (2.4), it follows that for all $k \in BV[0, T] \setminus \{0\}$,

$$(3.3) \quad E_x^{\text{anf}_q}[\Psi_u(\mathcal{Z}_k(x, \cdot))] = \exp\left\{\frac{i}{2q}\|uk\|_2^2\right\}.$$

Thus, using equations (2.9) and (3.3), we see that the L_1 analytic \mathcal{Z}_k -FFT, $T_{q, k}^{(1)}$, exists for all $q \in \mathbb{R} \setminus \{0\}$, and is given by

$$(3.4) \quad T_{q, k}^{(1)}(\Psi_u)(y) = \Psi_u(y)E_x^{\text{anf}_q}[\Psi_u(\mathcal{Z}_k(x, \cdot))] = \Psi_u^{q, u, k}(y)$$

for s-a.e. $y \in C_0[0, T]$. From equation (3.4), we also see that $T_{q, k}^{(1)} : \mathcal{E}(C_0[0, T]) \rightarrow \mathcal{E}(C_0[0, T])$ is well defined.

Next, using (2.10) with F and G replaced with Ψ_u and Ψ_v , it follows that for all real $q \in \mathbb{R} \setminus \{0\}$ and $g_1, g_2, h_3, h_4 \in L_2[0, T]$, the CTO of Ψ_u and Ψ_v , $(\Psi_u * \Psi_v)_q^{(g_1, g_2; h_3, h_4)}$, exists and is given by

$$(3.5) \quad (\Psi_u * \Psi_v)_q^{(g_1, g_2; h_1, h_2)}(y) = \exp\left\{\langle ug_1 + vg_2, y \rangle + \frac{i}{2q}\|uh_1 + vh_2\|_2^2\right\}$$

for s-a.e. $y \in C_0[0, T]$. Also, the functional $(\Psi_u * \Psi_v)_q^{(g_1, g_2; h_3, h_4)} \equiv \Psi_{ug_1 + vg_2}^{q, uh_1 + vh_2, 1}$ is an element of $\mathcal{E}(C_0[0, T])$.

Using (3.5) and applying the techniques similar to those used in (3.4), one can see that for s-a.e. $y \in C_0[0, T]$,

$$\begin{aligned} & T_{q, k}((\Phi_u * \Phi_v)_q^{(g_1, g_2; h_1, h_2)})(y) \\ &= \exp\left\{\langle ug_1 + vg_2, y \rangle + \frac{i}{2q}\|uh_1 + vh_2\|_2^2 + \frac{i}{2q}\|(ug_1 + vg_2)k\|_2^2\right\} \\ &= \exp\left\{\langle ug_1 + vg_2, y \rangle + \frac{i}{q} \int_0^T u(t)v(t)(h_1(t)h_2(t) + g_1(t)g_2(t)k^2(t))dt \right. \\ &\quad \left. + \frac{i}{2q} \int_0^T u^2(t)(h_1^2(t) + g_1^2(t)k^2(t))dt + \frac{i}{2q} \int_0^T v^2(t)(h_2^2(t) + g_2^2(t)k^2(t))dt\right\}. \end{aligned}$$

In order to obtain an equation similar to (1.1), one may put the condition that

$$(3.6) \quad h_1(t)h_2(t) + g_1(t)g_2(t)k^2(t) = 0 \quad m_L\text{-a.e. on } [0, T].$$

Then we can expect the following equation:

$$(3.7) \quad T_{q, k}((\Phi_u * \Phi_v)_q^{(g_1, g_2; h_1, h_2)})(y) = T_{q, \mathbf{s}_1}(\Phi_u)(\mathcal{Z}_{g_1}(y, \cdot))T_{q, \mathbf{s}_2}(\Phi_v)(\mathcal{Z}_{g_2}(y, \cdot))$$

for s-a.e. $y \in C_0[0, T]$, where $\mathbf{s}_i (i = 1, 2)$ is the function of bounded variation on $[0, T]$ such that $\mathbf{s}_i^2(t) = g_i^2(t)k^2(t) + h_i^2(t)$ for m_L -a.e. on $[0, T]$.

On the other hand, using (3.4) and (2.10), we also obtain that for s-a.e. $y \in C_0[0, T]$,

$$\begin{aligned} & (T_{q,k_1}(F) * T_{q,k_2}(G))_q^{(g_1, g_2; h_3, h_4)}(y) \\ &= \exp \left\{ \langle ug_1 + vg_2, y \rangle + \frac{i}{q} \int_0^T u(t)v(t)h_3(t)h_4(t)dt \right. \\ & \quad \left. + \frac{i}{2q} \int_0^T u^2(t)(h_3^2(t) + k_1^2(t)) + \frac{i}{2q} \int_0^T v^2(t)(h_4^2(t) + k_2^2(t)) \right\}, \end{aligned}$$

and under the condition

$$(3.8) \quad m_L(\text{supp}(h_3) \cap \text{supp}(h_4)) = 0.$$

it follows that

$$(3.9) \quad (T_{q,k_1}(\Phi_u) * T_{q,k_2}(\Phi_v))_q^{(g_1, g_2; h_3, h_4)}(y) = T_{q, \mathbf{s}_3}(\Phi_u)(\mathcal{Z}_{g_1}(y, \cdot))T_{q, \mathbf{s}_4}(\Phi_v)(\mathcal{Z}_{g_2}(y, \cdot))$$

for s-a.e. $y \in C_0[0, T]$, where \mathbf{s}_3 and \mathbf{s}_4 are the functions of bounded variation on $[0, T]$ such that $\mathbf{s}_3^2(t) = h_3^2(t) + k_1^2(t)$ and $\mathbf{s}_4^2(t) = h_4^2(t) + k_2^2(t)$.

In Section 5 below, we establish the relationships appearing in (3.7) and (3.9) for general functionals F and G on Wiener space. Equations (3.6) and (3.8) play key roles in our main theorems (see Theorems 5.3 and 5.5 below) in this paper.

4. ROTATION PROPERTIES OF GAUSSIAN PROCESSES

The essential purpose of this section is to establish rotation properties of Gaussian processes on the product Wiener spaces $C_0^2[0, T]$ and $C_0^3[0, T]$. For $h_1, h_2 \in BV[0, T]$ with $\|h_j\|_2 > 0$, $j \in \{1, 2\}$, let \mathcal{Z}_{h_1} and \mathcal{Z}_{h_2} be the Gaussian processes given by (2.1) with h replaced with h_1 and h_2 respectively. Then the process

$$\mathfrak{Z}_{h_1, h_2} : C_0[0, T] \times C_0[0, T] \times [0, T] \rightarrow \mathbb{R}$$

given by

$$\mathfrak{Z}_{h_1, h_2}(x_1, x_2, t) = \mathcal{Z}_{h_1}(x_1, t) + \mathcal{Z}_{h_2}(x_2, t)$$

is also a Gaussian process with mean zero and covariance function

$$\begin{aligned} & E_{x_1} [E_{x_2} [\mathfrak{Z}_{h_1, h_2}(x_1, x_2, s) \mathfrak{Z}_{h_1, h_2}(x_1, x_2, t)]] \\ &= \beta_{h_1}(\min\{s, t\}) + \beta_{h_2}(\min\{s, t\}) \\ &= \mathbf{v}_{h_1, h_2}(\min\{s, t\}). \end{aligned}$$

On the other hand, let h_1 and h_2 be elements of $BV[0, T]$. Then there exists a function $\mathbf{s} \in BV[0, T]$ such that

$$(4.1) \quad \mathbf{s}^2(t) = h_1^2(t) + h_2^2(t)$$

for m_L -a.e. $t \in [0, T]$, where m_L denotes the Lebesgue on $[0, T]$. Note that the function ' \mathbf{s} ' satisfying (4.1) is not unique. We will use the symbol $\mathbf{s}(h_1, h_2)$ for the functions ' \mathbf{s} ' that satisfy (4.1) above.

Given $h_1, h_2 \in BV[0, T]$, we consider the stochastic process $\mathcal{Z}_{\mathbf{s}(h_1, h_2)}$. Then $\mathcal{Z}_{\mathbf{s}(h_1, h_2)}$ is a Gaussian process with mean zero and covariance

$$\begin{aligned} E_x [\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, s) \mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, t)] \\ &= \int_0^{\min\{s, t\}} s^2(h_1, h_2)(u) db(u) = \int_0^{\min\{s, t\}} (h_1^2(u) + h_2^2(u)) db(u) \\ &= \beta_{h_1}(\min\{s, t\}) + \beta_{h_2}(\min\{s, t\}) = \mathbf{v}_{h_1, h_2}(\min\{s, t\}). \end{aligned}$$

From these facts, one can see that \mathfrak{Z}_{h_1, h_2} and $\mathcal{Z}_{\mathbf{s}(h_1, h_2)}$ have the same distribution and that for any random variable F on $C_0[0, T]$,

$$(4.2) \quad E_{x_1} \left[E_{x_2} \left[F \left(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot) \right) \right] \right] \stackrel{*}{=} E_x \left[F \left(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot) \right) \right],$$

where by $\stackrel{*}{=}$ we mean that if either side exists, both sides exist and equality holds.

4.1. A rotation property of Gaussian processes on $C_0^2[0, T]$. The following lemma will be very useful in our main theorem.

Lemma 4.1. *Given functions h_1, h_2, h_3 and h_4 in $L_2[0, T] \setminus \{0\}$, let the two stochastic processes \mathfrak{Z}_{h_1, h_2} and \mathfrak{Z}_{h_3, h_4} on $C_0^2[0, T] \times [0, T]$ be given by*

$$(4.3) \quad \mathfrak{Z}_{h_1, h_2}(x_1, x_2, t) = \mathcal{Z}_{h_1}(x_1, t) + \mathcal{Z}_{h_2}(x_2, t)$$

and

$$(4.4) \quad \mathfrak{Z}_{h_3, h_4}(x_1, x_2, t) = \mathcal{Z}_{h_3}(x_1, t) + \mathcal{Z}_{h_4}(x_2, t),$$

respectively. Then the following assertions are equivalent.

- (i) \mathfrak{Z}_{h_1, h_2} and \mathfrak{Z}_{h_3, h_4} are independent processes,
- (ii) $h_1 h_3 + h_2 h_4 = 0$.

Proof. Since the processes \mathfrak{Z}_{h_1, h_2} and \mathfrak{Z}_{h_3, h_4} are Gaussian, we know that \mathfrak{Z}_{h_1, h_2} and \mathfrak{Z}_{h_3, h_4} are independent if and only if

$$E_{x_2} [E_{x_1} [\mathfrak{Z}_{h_1, h_2}(x_1, x_2, s) \mathfrak{Z}_{h_3, h_4}(x_1, x_2, t)]] = 0$$

for all $s, t \in [0, T]$. But using the Fubini theorem and equation (2.2), we have

$$\begin{aligned} &E_{x_2} [E_{x_1} [\mathfrak{Z}_{h_1, h_2}(x_1, x_2, s) \mathfrak{Z}_{h_3, h_4}(x_1, x_2, t)]] \\ &= E_{x_2} [E_{x_1} [(\mathcal{Z}_{h_1}(x_1, s) \mathcal{Z}_{h_3}(x_1, t) + \mathcal{Z}_{h_1}(x_1, s) \mathcal{Z}_{h_4}(x_2, t) \\ &\quad + \mathcal{Z}_{h_2}(x_2, s) \mathcal{Z}_{h_3}(x_1, t) + \mathcal{Z}_{h_2}(x_2, s) \mathcal{Z}_{h_4}(x_2, t))] \\ &= \int_0^{\min\{s, t\}} h_1(u) h_3(u) du + \int_0^{\min\{s, t\}} h_2(u) h_4(u) du \\ &= \int_0^{\min\{s, t\}} (h_1(u) h_3(u) + h_2(u) h_4(u)) du. \end{aligned}$$

From this we can obtain the desired result. \square

Theorem 4.2. *Let h_1, h_2, h_3 and h_4 be nonzero elements of $BV[0, T]$ with $h_1 h_3 + h_2 h_4 = 0$, and let $\mathbf{F} : C_0^2[0, T] \rightarrow \mathbb{C}$ be a $m \times m$ -integrable functional. Then*

$$\begin{aligned} &E_{x_1} [E_{x_2} [\mathbf{F}(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot), \mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_2, \cdot))] \\ &= E_y [E_x [\mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), \mathcal{Z}_{\mathbf{s}(h_3, h_4)}(y, \cdot))]]. \end{aligned}$$

Proof. Let the processes $\mathfrak{Z}_{h_1, h_2}, \mathfrak{Z}_{h_3, h_4} : C_0^2[0, T] \times [0, T] \rightarrow \mathbb{R}$ be given by equation (4.3) and (4.4) respectively. Since h_i 's are functions of bounded variation, for all $(x_1, x_2) \in C_0^2[0, T]$ the sample paths $\mathfrak{Z}_{h_1, h_2}(x_1, x_2, \cdot)$ and $\mathfrak{Z}_{h_3, h_4}(x_1, x_2, \cdot)$ of the processes are continuous functions on $[0, T]$. Let X_{h_1, h_2} and X_{h_3, h_4} be measurable transforms from $C_0^2[0, T]$ into $C_0^2[0, T]$ given by

$$X_{h_1, h_2}(x_1, x_2) = \mathfrak{Z}_{h_1, h_2}(x_1, x_2, \cdot)$$

and

$$X_{h_3, h_4}(x_1, x_2) = \mathfrak{Z}_{h_3, h_4}(x_1, x_2, \cdot)$$

respectively. Also let $P \equiv X_{h_1, h_2}(C_0^2[0, T])$ and $Q \equiv X_{h_3, h_4}(C_0^2[0, T])$ be the image spaces of the measurable transforms X_{h_1, h_2} and X_{h_3, h_4} respectively. For simplicity, let m^2 denote the product Wiener measure $m \times m$ on $C_0^2[0, T]$.

By Lemma 4.6, we see that \mathfrak{Z}_{h_1, h_2} and \mathfrak{Z}_{h_3, h_4} are independent processes on $C_0^2[0, T]$ and so X_{h_1, h_2} and X_{h_3, h_4} are independent measurable transforms. Thus, by the change of variables formula, the Fubini theorem and (4.2), it follows that

$$\begin{aligned} & E_{x_2}[E_{x_1}[\mathbf{F}(\mathfrak{Z}_{h_1, h_2}(x_1, x_2, \cdot), \mathfrak{Z}_{h_3, h_4}(x_1, x_2, \cdot))]] \\ &= \int_{C_0^2[0, T]} \mathbf{F}(X_{h_1, h_2}(x_1, x_2), X_{h_3, h_4}(x_1, x_2)) dm^2(x_1, x_2) \\ &= \int_{P \times Q} \mathbf{F}(z_1, z_2) d[(m^2 \circ X_{h_1, h_2}^{-1}) \times (m^2 \circ X_{h_3, h_4}^{-1})](z_1, z_2) \\ &= \int_Q \left[\int_P \mathbf{F}(z_1, z_2) d(m^2 \circ X_{h_1, h_2}^{-1})(z_1) \right] d(m^2 \circ X_{h_3, h_4}^{-1})(z_2) \\ &= \int_Q \left[\int_{C_0^2[0, T]} \mathbf{F}(X_{h_1, h_2}(x_1, x_2), z_2) dm^2(x_1, x_2) \right] d(m^2 \circ X_{h_3, h_4}^{-1})(z_2) \\ &= \int_Q \left[\int_{C_0^2[0, T]} \mathbf{F}(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot), z_2) dm^2(x_1, x_2) \right] d(m^2 \circ X_{h_3, h_4}^{-1})(z_2) \\ &= \int_Q \left[\int_{C_0[0, T]} \mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), z_2) dm(x) \right] d(m^2 \circ X_{h_3, h_4}^{-1})(z_2) \\ &= \int_{C_0[0, T]} \left[\int_Q \mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), z_2) d(m^2 \circ X_{h_3, h_4}^{-1})(z_2) \right] dm(x) \\ &= \int_{C_0[0, T]} \left[\int_{C_0^2[0, T]} \mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), X_{h_3, h_4}(x_1, x_2)) dm^2(x_1, x_2) \right] dm(x) \\ &= \int_{C_0[0, T]} \left[\int_{C_0^2[0, T]} \mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), \mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_2, \cdot)) dm^2(x_1, x_2) \right] dm(x) \\ &= \int_{C_0[0, T]} \left[\int_{C_0[0, T]} \mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), \mathcal{Z}_{\mathbf{s}(h_3, h_4)}(y, \cdot)) dm(y) \right] dm(x). \end{aligned}$$

Thus we obtain the desired result. \square

The following corollaries are very simple consequences of Theorem 4.2.

Corollary 4.3. *Let h_1 and h_2 be nonzero elements of $BV[0, T]$ and let $\mathbf{F} : C_0^2[0, T] \rightarrow \mathbb{C}$ be a m^2 -integrable functional. Then*

$$\begin{aligned} (4.5) \quad & E_{x_2}[E_{x_1}[\mathbf{F}(\mathcal{Z}_{h_1}(x_1, \cdot) - \mathcal{Z}_{h_2}(x_2, \cdot), \mathcal{Z}_{h_2}(x_1, \cdot) + \mathcal{Z}_{h_1}(x_2, \cdot))]] \\ &= E_y[E_x[\mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), \mathcal{Z}_{\mathbf{s}(h_1, h_2)}(y, \cdot))]]. \end{aligned}$$

Remark 4.4. For any function $\theta(\cdot)$ of bounded variation, choosing $h_1(t) = \cos \theta(t)$ and $h_2(t) = \sin \theta(t)$ on $[0, T]$ in equation (4.5) yields the main result in [1].

Corollary 4.5. Let h_1, h_2, h_3 and h_4 be as in Theorem 4.2. Let F and G be μ -integrable functionals. Then

$$(4.6) \quad \begin{aligned} & E_{x_1} [E_{x_2} [F(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot)) G(\mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_2, \cdot))]] \\ &= E_x [F(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot))] E_x [G(\mathcal{Z}_{\mathbf{s}(h_3, h_4)}(x, \cdot))] \end{aligned}$$

4.2. A rotation property of Gaussian processes on $C_0^3[0, T]$.

Lemma 4.6. Given functions h_1, h_2, h_3 and h_4 in $L_2[0, T] \setminus \{0\}$, let the two stochastic processes $\mathfrak{Z}_{h_1, h_2, 0}$ and $\mathfrak{Z}_{h_3, 0, h_4}$ on $C_0^3[0, T] \times [0, T]$ be given by

$$(4.7) \quad \mathfrak{W}_{h_1, h_2, 0}(x_1, x_2, x_3, t) = \mathcal{Z}_{h_1}(x_1, t) + \mathcal{Z}_{h_2}(x_2, t)$$

and

$$(4.8) \quad \mathfrak{W}_{h_3, 0, h_4}(x_1, x_2, x_3, t) = \mathcal{Z}_{h_3}(x_1, t) + \mathcal{Z}_{h_4}(x_3, t),$$

respectively. If $m_L(\text{supp}(h_1) \cap \text{supp}(h_3)) = 0$, then $\mathfrak{W}_{h_1, h_2, 0}$ and $\mathfrak{W}_{h_3, 0, h_4}$ are independent processes.

Remark 4.7. By the consistency property, the processes $\mathfrak{W}_{h_1, h_2, 0}$ and $\mathfrak{W}_{h_1, 0, h_3}$ can be considered as processes on $C_0^2[0, T] \times [0, T]$.

Proof of Lemma 4.6. Since the processes $\mathfrak{W}_{h_1, h_2, 0}$ and \mathfrak{W}_{h_3, h_4} are Gaussian, we know that $\mathfrak{W}_{h_1, h_2, 0}$ and $\mathfrak{W}_{h_3, 0, h_4}$ are independent if and only if

$$E_{x_3} [E_{x_2} [E_{x_1} [\mathfrak{W}_{h_1, h_2, 0}(x_1, x_2, x_3, s) \mathfrak{W}_{h_3, 0, h_4}(x_1, x_2, x_3, t)]]] = 0$$

for all $s, t \in [0, T]$. But using the Fubini theorem and equation (2.2), we have

$$\begin{aligned} & E_{x_3} [E_{x_2} [E_{x_1} [\mathfrak{W}_{h_1, h_2, 0}(x_1, x_2, x_3, s) \mathfrak{W}_{h_3, 0, h_4}(x_1, x_2, x_3, t)]]] \\ &= E_{x_3} [E_{x_2} [E_{x_1} [\left(\mathcal{Z}_{h_1}(x_1, s) \mathcal{Z}_{h_3}(x_1, t) + \mathcal{Z}_{h_1}(x_1, s) \mathcal{Z}_{h_4}(x_3, t) \right. \\ &\quad \left. + \mathcal{Z}_{h_2}(x_2, s) \mathcal{Z}_{h_3}(x_1, t) + \mathcal{Z}_{h_2}(x_2, s) \mathcal{Z}_{h_4}(x_3, t) \right)]]] \\ &= \int_0^{\min\{s, t\}} h_1(u) h_3(u) du \end{aligned}$$

From this we can obtain the desired result. \square

Theorem 4.8. Let h_1, h_2, h_3 and h_4 be nonzero elements of $BV[0, T]$ with

$$m_L(\text{supp}(h_1) \cap \text{supp}(h_3)) = 0,$$

and let $\mathbf{F} : C_0^2[0, T] \rightarrow \mathbb{C}$ be a m^2 -integrable functional. Then

$$\begin{aligned} & E_{x_3} [E_{x_2} [E_{x_1} [\mathbf{F}(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot), \mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_3, \cdot))]]] \\ &= E_y [E_x [\mathbf{F}(\mathcal{Z}_{\mathbf{s}(h_1, h_2)}(x, \cdot), \mathcal{Z}_{\mathbf{s}(h_3, h_4)}(y, \cdot))]]. \end{aligned}$$

Proof. Let the processes $\mathfrak{W}_{h_1, h_2, 0}, \mathfrak{W}_{h_3, 0, h_4} : C_0^3[0, T] \times [0, T] \rightarrow \mathbb{R}$ be given by equation (4.7) and (4.8) respectively. Since h_i 's are functions of bounded variation, for all $(x_1, x_2, x_3) \in C_0^3[0, T]$ the sample paths $\mathfrak{W}_{h_1, h_2, 0}(x_1, x_2, x_3, \cdot)$ and

$\mathfrak{W}_{h_3,0,h_4}(x_1, x_2, x_3, \cdot)$ of the processes are continuous functions on $[0, T]$. Let $Y_{h_1,h_2,0}$ and $Y_{h_3,0,h_4}$ be measurable transforms from $C_0^3[0, T]$ into $C_0^3[0, T]$ given by

$$Y_{h_1,h_2,0}(x_1, x_2, x_3) = \mathfrak{W}_{h_1,h_2,0}(x_1, x_2, x_3, \cdot)$$

and

$$Y_{h_3,0,h_4}(x_1, x_2, x_3) = \mathfrak{W}_{h_3,0,h_4}(x_1, x_2, x_3, \cdot)$$

respectively. Also let $M \equiv Y_{h_1,h_2,0}(C_0^3[0, T])$ and $N \equiv Y_{h_3,0,h_4}(C_0^3[0, T])$ be the image spaces of the measurable transforms $Y_{h_1,h_2,0}$ and $Y_{h_3,0,h_4}$ respectively. For simplicity, let m^3 denote the product Wiener measure $m \times m \times m$ on $C_0^3[0, T]$.

By Lemma 4.6, we see that $\mathfrak{W}_{h_1,h_2,0}$ and $\mathfrak{W}_{h_3,0,h_4}$ are independent processes on $C_0^3[0, T]$ and so $Y_{h_1,h_2,0}$ and $Y_{h_3,0,h_4}$ are independent measurable transforms. Thus, by the change of variables formula, the Fubini theorem, (4.2), and Remark 4.7 it follows that

$$\begin{aligned} & E_{x_3} \left[E_{x_2} \left[E_{x_1} \left[\mathbf{F} \left(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot), \mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_3, \cdot) \right) \right] \right] \right] \\ &= \int_{C_0^3[0, T]} \mathbf{F} \left(Y_{h_1,h_2,0}(x_1, x_2, x_3), Y_{h_3,0,h_4}(x_1, x_2, x_3) \right) dm^3(x_1, x_2, x_3) \\ &= \int_N \left[\int_M \mathbf{F}(w_1, w_2) d(m^3 \circ Y_{h_1,h_2,0}^{-1})(w_1) \right] d(m^3 \circ Y_{h_3,0,h_4}^{-1})(w_2) \\ &= \int_N \left[\int_{C_0^3[0, T]} \mathbf{F} \left(Y_{h_1,h_2,0}(x_1, x_2, x_3), w_2 \right) dm^3(x_1, x_2, x_3) \right] d(m^3 \circ Y_{h_3,0,h_4}^{-1})(w_2) \\ &= \int_N \left[\int_{C_0^2[0, T]} \mathbf{F} \left(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot), w_2 \right) dm^2(x_1, x_2) \right] d(m^2 \circ Y_{h_3,0,h_4}^{-1})(w_2) \\ &= \int_N \left[\int_{C_0[0, T]} \mathbf{F} \left(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x, \cdot), w_2 \right) dm(x) \right] d(m^3 \circ Y_{h_3,0,h_4}^{-1})(w_2) \\ &= \int_{C_0[0, T]} \left[\int_N \mathbf{F} \left(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x, \cdot), w_2 \right) d(m^3 \circ Y_{h_3,0,h_4}^{-1})(w_2) \right] dm(x) \\ &= \int_{C_0[0, T]} \left[\int_{C_0^3[0, T]} \mathbf{F} \left(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x, \cdot), Y_{h_3,0,h_4}(x_1, x_2, x_3) \right) dm^3(x_1, x_2, x_3) \right] dm(x) \\ &= \int_{C_0[0, T]} \left[\int_{C_0^2[0, T]} \mathbf{F} \left(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x, \cdot), \mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_3, \cdot) \right) dm^2(x_1, x_3) \right] dm(x) \\ &= \int_{C_0[0, T]} \left[\int_{C_0[0, T]} \mathbf{F} \left(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x, \cdot), \mathcal{Z}_{\mathbf{s}(h_3,h_4)}(y, \cdot) \right) dm(y) \right] dm(x). \end{aligned}$$

Thus we obtain the desired result. \square

The following corollary is a very simple consequence of Theorem 4.8.

Corollary 4.9. *Let h_1, h_2, h_3 and h_4 be as in Theorem 4.8. Let F and G be μ -integrable functionals. Then*

$$\begin{aligned} (4.9) \quad & E_{x_3} \left[E_{x_2} \left[E_{x_1} \left[F \left(\mathcal{Z}_{h_1}(x_1, \cdot) + \mathcal{Z}_{h_2}(x_2, \cdot) \right) G \left(\mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{h_4}(x_3, \cdot) \right) \right] \right] \right] \\ &= E_x \left[F \left(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x, \cdot) \right) \right] E_x \left[G \left(\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(x, \cdot) \right) \right] \end{aligned}$$

5. FOURIER–FEYNMAN TRANSFORMS AND CONVOLUTION TYPE OPERATIONS

In this section we establish the facts that the Fourier–Feynman transform of the convolution type operation is a product of the Fourier–Feynman transforms (Theorem 5.3 below) and that the convolution type operation of the Fourier–Feynman transforms is a product of the Fourier–Feynman transforms (Theorem 5.5 below).

5.1. Transforms of convolution type operations. It will be helpful to establish the following lemma before giving the first theorem in this section.

Lemma 5.1. *Let g_1, g_2, h_1, h_2 and k be nonzero functions in $BV[0, T]$. For each $j \in \{1, 2\}$, let $\mathbf{s}(g_j h_j, h_j)$ be the functions in $BV[0, T]$ satisfying equation (4.1) with h_1 and h_2 replaced with $g_j h_j$ and h_j . Also, let F and G be \mathbb{C} -valued scale invariant measurable functionals on $C_0[0, T]$ such that the analytic Wiener integrals $T_{\lambda, \mathbf{s}(g_1 k, h_1)}(F)(y)$, $T_{\lambda, \mathbf{s}(g_2 k, h_2)}(G)(y)$ and $(F * G)_{\lambda}^{(g_1, g_2; h_1, h_2)}(y)$ exist for every $\lambda \in \mathbb{C}$ and s-a.e. $y \in C_0[0, T]$. Furthermore assume that given $k \in BV[0, T] \setminus \{0\}$, the analytic transform of $(F * G)_{\lambda_1}^{(g_1, g_2; h_1, h_2)}$, $T_{\lambda_2, k}((F * G)_{\lambda_1}^{(g_1, g_2; h_1, h_2)})(y)$ exists for every $(\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$ and s-a.e. $y \in C_0[0, T]$. Suppose $g_1 g_2 k^2 + h_1 h_2 = 0$. Then for each $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0[0, T]$,*

$$(5.1) \quad \begin{aligned} & T_{\lambda, k}((F * G)_{\lambda}^{(g_1, g_2; h_1, h_2)})(y) \\ &= T_{\lambda, \mathbf{s}(g_1 k, h_1)}(F)(\mathcal{Z}_{g_1}(y, \cdot)) T_{\lambda, \mathbf{s}(g_2 k, h_2)}(G)(\mathcal{Z}_{g_2}(y, \cdot)). \end{aligned}$$

Remark 5.2. (Comments on the assumptions in Lemma 5.1)

Let a function $y \in C_0[0, T]$ be given. For $(\lambda_1, \rho_2) \in \mathbb{C}_+ \times (0, +\infty)$, let

$$\begin{aligned} J_{\lambda_1}^*(k; \rho_2) &:= T_{k, \rho_2}((F * G)_{\lambda_1}^{(g_1, g_2; h_1, h_2)})(y) \\ &= E_{x_2}[(F * G)_{\lambda_1}^{(g_1, g_2; h_1, h_2)}(y + \rho_2^{-1/2} \mathcal{Z}_k(x_2, \cdot))] \\ &= E_{x_2}[E_{x_1}^{\text{anw}_{\lambda_1}}[F(\mathcal{Z}_{g_1}(y + \rho_2^{-1/2} \mathcal{Z}_k(x_2, \cdot)) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\ &\quad \times G(\mathcal{Z}_{g_2}(y + \rho_2^{-1/2} \mathcal{Z}_k(x_2, \cdot)) + \mathcal{Z}_{h_2}(x_1, \cdot))]] \\ &= E_{x_2}[E_{x_1}^{\text{anw}_{\lambda_1}}[F(\mathcal{Z}_{g_1}(y, \cdot) + \rho_2^{-1/2} \mathcal{Z}_{g_1 k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\ &\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \rho_2^{-1/2} \mathcal{Z}_{g_2 k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot))]]], \end{aligned}$$

for $(\rho_1, \lambda_2) \in (0, +\infty) \times \mathbb{C}_+$, let

$$\begin{aligned} J_{\rho_1}^*(k; \lambda_2) &:= T_{k, \lambda_2}((F * G)_{\rho_1}^{(g_1, g_2; h_1, h_2)})(y) \\ &= E_{x_2}^{\text{anw}_{\lambda_2}}[(F * G)_{\rho_1}^{(g_1, g_2; h_1, h_2)}(y + \mathcal{Z}_k(x_2, \cdot))] \\ &= E_{x_2}^{\text{anw}_{\lambda_2}}[E_{x_1}[F(\mathcal{Z}_{g_1}(y + \mathcal{Z}_k(x_2, \cdot)) + \rho_1^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot)) \\ &\quad \times G(\mathcal{Z}_{g_2}(y + \mathcal{Z}_k(x_2, \cdot)) + \rho_1^{-1/2} \mathcal{Z}_{h_2}(x_1, \cdot))]] \\ &= E_{x_2}^{\text{anw}_{\lambda_2}}[E_{x_1}[F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{g_1 k}(x_2, \cdot) + \rho_1^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot)) \\ &\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{g_2 k}(x_2, \cdot) + \rho_1^{-1/2} \mathcal{Z}_{h_2}(x_1, \cdot))]]], \end{aligned}$$

and for $(\rho_1, \rho_2) \in (0, +\infty) \times (0, +\infty)$, let

$$\begin{aligned}
J_{(F,G)}(k, k; \rho_1, \rho_2) &= T_{k, \rho_2}((F * G)_{\rho_1}^{(g_1, g_2; h_1, h_2)})(y) \\
&= E_{x_2}[(F * G)_{\rho_1}^{(g_1, g_2; h_1, h_2)}(y + \rho_2^{-1/2} \mathcal{Z}_k(x_2, \cdot))] \\
&= E_{x_2}[E_{x_1}[F(\mathcal{Z}_{g_1}(y + \rho_2^{-1/2} \mathcal{Z}_k(x_2, \cdot)) + \rho_1^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y + \rho_2^{-1/2} \mathcal{Z}_k(x_2, \cdot)) + \rho_1^{-1/2} \mathcal{Z}_{h_2}(x_1, \cdot))] \\
&= E_{x_2}[E_{x_1}[F(\mathcal{Z}_{g_1}(y, \cdot) + \rho_2^{-1/2} \mathcal{Z}_{g_1 k}(x_2, \cdot) + \rho_1^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \rho_2^{-1/2} \mathcal{Z}_{g_2 k}(x_2, \cdot) + \rho_1^{-1/2} \mathcal{Z}_{h_2}(x_1, \cdot))]],
\end{aligned}$$

Also, let $J_{\lambda_1}^*(k; \lambda_2)$, $\lambda_2 \in \mathbb{C}_+$, denote the analytic continuation of $J_{\lambda_1}^*(k; \rho_2)$, let $J_{\lambda_1}^*(k; \lambda_1)$, $\lambda_1 \in \mathbb{C}_+$, denote the analytic continuation of $J_{\lambda_2}^*(h_1; \rho_1)$, and let $J_{(F,G)}^{**}(k, k; \cdot, \cdot)$ on $\mathbb{C}_+ \times \mathbb{C}_+$, denote the analytic continuation of $J_{(F,G)}(k, k; \rho_1, \rho_2)$.

From the assumptions in Lemma 5.1, one can see that the three analytic Wiener integrals $J_{\lambda_1}^*(k; \lambda_2)$, $J_{\lambda_1}^*(k; \lambda_1)$, and $J_{(F,G)}^{**}(k, k; \lambda_1, \lambda_2)$ all exist, and

$$(5.2) \quad J_{\lambda_1}^*(k; \lambda_2) = J_{\lambda_1}^*(k; \lambda_1) = J_{(F,G)}^{**}(k, k; \lambda_1, \lambda_2)$$

for all $(\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$.

Proof of Lemma 5.1. In view of equations (2.8) and (2.10), we first note that the existences of the analytic Wiener integrals

$$T_{\lambda, \mathbf{s}(g_1 h_1, h_1)}(F)(y), \quad T_{\lambda, \mathbf{s}(g_2 h_2, h_2)}(G)(y), \quad (F * G)_{\lambda}^{(g_1, g_2; h_1, h_2)}(y)$$

and

$$T_{\lambda_2, k}((F * G)_{\lambda_1}^{(g_1, g_2; h_1, h_2)})(y)$$

guarantee that the five analytic Wiener integrals

$$\begin{aligned}
&\text{(i)} \quad E_x[F(y + \lambda^{-1/2} \mathcal{Z}_{\mathbf{s}(g_1 k, h_1)}(x, \cdot))], \\
&\text{(ii)} \quad E_x[F(y + \lambda^{-1/2} \mathcal{Z}_{\mathbf{s}(g_2 k, h_2)}(x, \cdot))], \\
&\text{(iii)} \quad E_x[F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_1}(x, \cdot))G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_2}(x, \cdot))], \\
&\text{(iv)} \quad E_{x_2}[E_{x_1}[F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda_2^{-1/2} \mathcal{Z}_{g_1 k}(x_2, \cdot) + \lambda_1^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda_2^{-1/2} \mathcal{Z}_{g_2 k}(x_2, \cdot) + \lambda_1^{-1/2} \mathcal{Z}_{h_2}(x_1, \cdot))]],
\end{aligned}$$

and

$$\begin{aligned}
&\text{(v)} \quad E_{x_2}[E_{x_1}^{\text{an} \zeta_1}[F(\mathcal{Z}_{g_1}(y, \cdot) + \zeta_2^{-1/2} \mathcal{Z}_{g_1 k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \zeta_2^{-1/2} \mathcal{Z}_{g_2 k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot))]]
\end{aligned}$$

all exist for any $\lambda > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\zeta_1 \in \mathbb{C}_+$, $\zeta_2 > 0$, and s-a.e. $y \in C_0[0, T]$.

Next, the existence of the analytic Wiener integral

$$\begin{aligned}
&\mathfrak{J}(\lambda_1, \lambda_2) \equiv T_{\lambda_2, k}((F * G)_{\lambda_1}^{(g_1, g_2; h_1, h_2)}) \\
(5.3) \quad &= E_{x_2}^{\text{an} w \lambda_2}[E_{x_1}^{\text{an} w \lambda_1}[F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{g_1 k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{g_2 k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot))]]
\end{aligned}$$

also ensure that the analytic Wiener integral

$$\begin{aligned} \mathfrak{J}(\lambda, \lambda) &= E_{x_2}^{anw\lambda} \left[E_{x_1}^{anw\lambda} \left[F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{g_1k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \right. \right. \\ &\quad \left. \left. \times G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{g_2k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot)) \right] \right] \end{aligned}$$

is well-defined for all $\lambda \in \mathbb{C}_+$. In equation (5.3) above, by the observation in Remark 5.2, $\mathfrak{J}(\lambda_1, \lambda_2)$ means the three analytic function space integrals in equation (5.2).

On the other hand, using the Fubini theorem and (4.6), it follows that for all $\lambda > 0$ and s-a.e. $y \in C_0[0, T]$,

$$\begin{aligned} T_{\lambda, k}((F * G)_{\lambda}^{(g_1, g_2; h_1, h_2)})(y) &\equiv \mathfrak{J}(\lambda, \lambda) \\ &= E_{x_2} \left[E_{x_1} \left[F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{g_1k}(x_2, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_1}(x_1, \cdot)) \right. \right. \\ &\quad \left. \left. \times G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{g_2k}(x_2, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_2}(x_1, \cdot)) \right] \right] \\ &= E_{x_2} \left[E_{x_1} \left[F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} [\mathcal{Z}_{g_1k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)]) \right. \right. \\ &\quad \left. \left. \times G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} [\mathcal{Z}_{g_2k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot)]) \right] \right] \\ &= E_x \left[F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{s(g_1k, h_1)}(x, \cdot)) \right] \\ &\quad \times E_x \left[G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{s(g_2k, h_2)}(x, \cdot)) \right] \\ &= T_{\lambda, s(g_1k, h_1)}(F)(\mathcal{Z}_{g_1}(y, \cdot)) T_{\lambda, s(g_2k, h_2)}(G)(\mathcal{Z}_{g_2}(y, \cdot)). \end{aligned}$$

We now use the analytic continuation to obtain our desired conclusion. \square

Theorem 5.3. *Let $g_1, g_2, h_1, h_2, k, s(g_1h_1, h_1)$ and $s(g_2h_2, h_2)$ be as in Lemma 5.1. Let q be a nonzero real number and let F and G be \mathbb{C} -valued scale invariant measurable functionals on $C_0[0, T]$ such that the L_1 analytic Feynman integrals $T_{q, s(g_1k, h_1)}(F)(y)$, $T_{q, s(g_2k, h_2)}(G)(y)$ and $(F * G)_q^{(g_1, g_2; h_1, h_2)}(y)$ exist for s-a.e. $y \in C_0[0, T]$. Furthermore assume that given $k \in BV[0, T] \setminus \{0\}$, the analytic FFT of $(F * G)_q^{(g_1, g_2; h_1, h_2)}$, $T_{q, k}((F * G)_q^{(g_1, g_2; h_1, h_2)})(y)$ exists for s-a.e. $y \in C_0[0, T]$. Suppose*

$$g_1g_2k^2 + h_1h_2 = 0.$$

Then for s-a.e. $y \in C_0[0, T]$,

$$\begin{aligned} (5.4) \quad T_{q, k}((F * G)_q^{(g_1, g_2; h_1, h_2)})(y) \\ = T_{q, s(g_1k, h_1)}(F)(\mathcal{Z}_{g_1}(y, \cdot)) T_{q, s(g_2k, h_2)}(G)(\mathcal{Z}_{g_2}(y, \cdot)). \end{aligned}$$

Remark 5.4. (Comments on the assumptions in Theorem 5.3)

Before giving the proof of Theorem 5.3, we will emphasize the following assertions:

(i) The existence conditions for

$$T_{q, s(g_1k, h_1)}(F), T_{q, s(g_2k, h_2)}(G) \text{ and } (F * G)_q^{(g_1, g_2; h_1, h_2)}$$

say that

$$T_{\lambda, s(g_1k, h_1)}(F)(y), T_{\lambda, s(g_2k, h_2)}(G)(y) \text{ and } (F * G)_{\lambda}^{(g_1, g_2; h_1, h_2)}(y)$$

all exist for all $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0[0, T]$.

(ii) The existence conditions for $(F * G)_q^{(g_1, g_2; h_1, h_2)}$ and $T_{q, k}((F * G)_q^{(g_1, g_2; h_1, h_2)})$ say that

- $T_{\lambda,k}((F * G)_q^{(g_1,g_2;h_1,h_2)})(y)$ exists for every $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0[0, T]$; and
- $T_{\lambda_2,k}((F * G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)})(y)$ exists for every $(\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$ and s-a.e. $y \in C_0[0, T]$.

Thus the assumptions in Theorem 5.3 involve the assumptions in Lemma 5.1.

Proof of Theorem 5.3. To obtain equation (5.4), one may establish that

$$\begin{aligned}
& T_{q,k}((F * G)_q^{(g_1,g_2;h_1,h_2)})(y) \\
&= \lim_{\substack{\lambda_2 \rightarrow -iq \\ \lambda_2 \in \mathbb{C}_+}} E_{x_2}^{\text{an}w\lambda_2} [(F * G)_q^{(g_1,g_2;h_1,h_2)}(y + \mathcal{Z}_k(x_2, \cdot))] \\
&= \lim_{\substack{\lambda_1, \lambda_2 \rightarrow -iq \\ \lambda_1, \lambda_2 \in \mathbb{C}_+}} E_{x_2}^{\text{an}w\lambda_2} [E_{x_1}^{\text{an}w\lambda_1} [F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{g_1k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{g_2k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot))]]], \\
&= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} E_x^{\text{an}w\lambda} [F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x, \cdot))] \\
&\quad \times \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} E_x^{\text{an}w\lambda} [G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x, \cdot))] \\
&= T_{q,\mathbf{s}(g_1k,h_1)}(F)(\mathcal{Z}_g(y, \cdot)) T_{q,\mathbf{s}(g_2k,h_2)}(G)(\mathcal{Z}_g(y, \cdot)).
\end{aligned}$$

But, as shown in the proof of Lemma 5.1, the assertions in Remark 5.4 that the analytic Wiener integrals

$$\begin{aligned}
T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y) &= E_x^{\text{an}w\lambda} [F(y + \mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x, \cdot))] \\
T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y) &= E_x^{\text{an}w\lambda} [G(y + \mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x, \cdot))]
\end{aligned}$$

and

$$\begin{aligned}
& (F * G)_\lambda^{(g_1,g_2;h_1,h_2)}(y) \\
&= E_x^{\text{an}w\lambda} [F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{h_1}(x, \cdot)) G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{h_2}(x, \cdot))]
\end{aligned}$$

exist for every $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0[0, T]$, and the analytic Wiener integral

$$\begin{aligned}
& T_{\lambda_2,k}((F * G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)})(y) \\
&= E_{x_2}^{\text{an}w\lambda_2} [E_{x_1}^{\text{an}w\lambda_1} [F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{g_1k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{g_2k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot))]]
\end{aligned}$$

exists for every $(\lambda_1, \lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$, say the fact that $T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y)$ and $T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y)$ are analytic on \mathbb{C}_+ , as functions of λ , and $T_{\lambda_2,k}((F * G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)})(y)$ is analytic on $\mathbb{C}_+ \times \mathbb{C}_+$, as a function of (λ_1, λ_2) . Thus, to establish equation (5.4),

it will suffice to show that

$$\begin{aligned}
& T_{q,k}((F * G)^{(g_1, g_2; h_1, h_2)})(y) \\
&= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} E_{x_2}^{\text{an}w\lambda} [E_{x_1}^{\text{an}w\lambda} [F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{g_1 k}(x_2, \cdot) + \mathcal{Z}_{h_1}(x_1, \cdot)) \\
&\quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{g_2 k}(x_2, \cdot) + \mathcal{Z}_{h_2}(x_1, \cdot))]]], \\
&= \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} E_x^{\text{an}w\lambda} [F(\mathcal{Z}_{g_1}(y, \cdot) + \mathcal{Z}_{\mathbf{s}(g_1 k, h_1)}(x, \cdot))] \\
&\quad \times \lim_{\substack{\lambda \rightarrow -iq \\ \lambda \in \mathbb{C}_+}} E_x^{\text{an}w\lambda} [G(\mathcal{Z}_{g_2}(y, \cdot) + \mathcal{Z}_{\mathbf{s}(g_2 k, h_2)}(x, \cdot))] \\
&= T_{q, \mathbf{s}(g_1 k, h_1)}(F)(\mathcal{Z}_{g_1}(y, \cdot)) T_{q, \mathbf{s}(g_2 k, h_2)}(G)(\mathcal{Z}_{g_2}(y, \cdot)).
\end{aligned}$$

But it follows from equation (5.1) and the analytic continuation. \square

5.2. Convolution type operations of transforms. In our second theorem we establish the fact that the convolution type operation of the Fourier–Feynman transforms is a product of the Fourier–Feynman transforms.

Theorem 5.5. *Let g_1, g_2, k_1, k_2, h_3 and h_4 be nonzero functions in $BV[0, T]$. Let $\mathbf{s}(h_3, k_1)$ and $\mathbf{s}(h_4, k_2)$ be given as in equation (4.1). Also, let F and G be \mathbb{C} -valued scale invariant measurable functionals on $C_0[0, T]$ such that given a real $q \in \mathbb{R} \setminus \{0\}$, the analytic FFTs on $C_0[0, T]$ such that $T_{q, k_1}(F)$, $T_{q, k_2}(G)$, $T_{q, \mathbf{s}(g_1 k, h_1)}(F)$ and $T_{q, \mathbf{s}(g_2 k, h_2)}(G)$ exist for s -a.e. $y \in C_0[0, T]$. Furthermore assume that the convolution type operation $(T_{q, k_1}(F) * T_{q, k_2}(G))_{\lambda_3}^{(g_1, g_2; h_3, h_4)}$ exists for s -a.e. $y \in C_0[0, T]$. Suppose*

$$(5.5) \quad m_L(\text{supp}(h_3) \cap \text{supp}(h_4)) = 0.$$

Then for given $q \in \mathbb{R} \setminus \{0\}$ and s -a.e. $y \in C_0[0, T]$,

$$\begin{aligned}
(5.6) \quad & (T_{q, k_1}(F) * T_{q, k_2}(G))_q^{(g_1, g_2; h_3, h_4)}(y) \\
&= T_{q, \mathbf{s}(h_3, k_1)}(F)(\mathcal{Z}_{g_1}(y, \cdot)) T_{q, \mathbf{s}(h_4, k_2)}(G)(\mathcal{Z}_{g_2}(y, \cdot)).
\end{aligned}$$

Proof. By similar arguments in Remark 5.4 and 5.2, the following analytic continuation of the seven Wiener integrals

$$\begin{aligned}
J_1(\rho_1, \rho_2, \rho_3) &= (T_{\rho_1, k_1}(F) * T_{\rho_2, k_2}(G))_{\rho_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_1, \rho_2, \rho_3 \in (0, +\infty) \\
J_2(\rho_2, \rho_3; \lambda_1) &= (T_{\lambda_1, k_1}(F) * T_{\rho_2, k_2}(G))_{\rho_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_2, \rho_3 \in (0, +\infty), \lambda_1 \in \mathbb{C}_+ \\
J_3(\rho_1, \rho_3; \lambda_2) &= (T_{\rho_1, k_1}(F) * T_{\lambda_2, k_2}(G))_{\rho_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_1, \rho_3 \in (0, +\infty), \lambda_2 \in \mathbb{C}_+ \\
J_4(\rho_1, \rho_2; \lambda_3) &= (T_{\rho_1, k_1}(F) * T_{\rho_2, k_2}(G))_{\lambda_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_1, \rho_2 \in (0, +\infty), \lambda_3 \in \mathbb{C}_+ \\
J_5(\rho_3; \lambda_1, \lambda_2) &= (T_{\lambda_1, k_1}(F) * T_{\lambda_2, k_2}(G))_{\rho_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_3 \in (0, +\infty), \lambda_1, \lambda_2 \in \mathbb{C}_+ \\
J_6(\rho_2; \lambda_1, \lambda_3) &= (T_{\lambda_1, k_1}(F) * T_{\rho_2, k_2}(G))_{\lambda_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_2 \in (0, +\infty), \lambda_1, \lambda_3 \in \mathbb{C}_+
\end{aligned}$$

and

$$J_7(\rho_1; \lambda_2, \lambda_3) = (T_{\rho_1, k_1}(F) * T_{\lambda_2, k_2}(G))_{\lambda_3}^{(g_1, g_2; h_3, h_4)}, \quad \rho_1 \in (0, +\infty), \lambda_2, \lambda_3 \in \mathbb{C}_+$$

all exist and have the same analytic continuation (analytic Wiener integral on $C_0^3[0, T]$)

$$J^*(\lambda_1, \lambda_2, \lambda_3) = (T_{\lambda_1, k_1}(F) * T_{\lambda_2, k_2}(G))_{\lambda_3}^{(g_1, g_2; h_3, h_4)}, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}_+.$$

Thus, by similar arguments as in the proofs of Lemma 5.1 and Theorem 5.3, it will suffice to show that equation (5.6) holds for all $\lambda > 0$ and s-a.e. $y \in C_0[0, T]$.

Using the Fubini theorem and applying equation (4.9) with the condition (5.5), it follows that for all $\lambda > 0$ and s-a.e. $y \in C_0[0, T]$,

$$\begin{aligned} & (T_{\lambda, k_1}(F) * T_{\lambda, k_2}(G))_{\lambda}^{(g_1, g_2; h_3, h_4)}(y) \\ &= E_{x_1} [T_{\lambda, k_1}(F)(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_3}(x_1, \cdot)) \\ & \quad \times T_{\lambda, k_2}(G)(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_4}(x_1, \cdot))] \\ &= E_{x_1} [E_{x_2} [F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_3}(x_1, \cdot) + \lambda^{-1/2} \mathcal{Z}_{k_1}(x_2, \cdot))] \\ & \quad \times E_{x_3} [G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{h_4}(x_1, \cdot) + \lambda^{-1/2} \mathcal{Z}_{k_2}(x_3, \cdot))]] \\ &= E_{x_1} [E_{x_2} [E_{x_3} [F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} (\mathcal{Z}_{h_3}(x_1, \cdot) + \mathcal{Z}_{k_1}(x_2, \cdot))) \\ & \quad \times G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} (\mathcal{Z}_{h_4}(x_1, \cdot) + \mathcal{Z}_{k_2}(x_3, \cdot))]]]] \\ &= E_x [F(\mathcal{Z}_{g_1}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{\mathbf{s}(h_3, k_1)}(x, \cdot))] \\ & \quad \times E_x [G(\mathcal{Z}_{g_2}(y, \cdot) + \lambda^{-1/2} \mathcal{Z}_{\mathbf{s}(h_4, k_2)}(x, \cdot))] \\ &= T_{\lambda, \mathbf{s}(h_3, k_1)}(F)(\mathcal{Z}_{g_1}(y, \cdot)) T_{\lambda, \mathbf{s}(h_4, k_2)}(G)(\mathcal{Z}_{g_2}(y, \cdot)) \end{aligned}$$

as desired. \square

6. FURTHER RESULT AND EXAMPLES

The result in Theorems 5.3 and 5.5 above can be applied to many large classes of functionals on $C_0[0, T]$. These classes of functionals are appeared in [3, 5, 9, 10, 11, 14, 19, 21].

In Theorem 5.3, we established that an analytic FFT with respect to a Gaussian process of a convolution type operation of functionals on $C_0[0, T]$ is a product of analytic FFTs with respect to a Gaussian processes, and in Theorem 5.5, we established that a convolution type operation of analytic FFTs with respect to a Gaussian processes is a product of analytic FFTs with respect to a Gaussian processes.

Here we have the following question: how to relate the two results in Theorems 5.3 and 5.5, i.e., how to find the conditions on the transforms and convolution type operations in the next equation?

$$(6.1) \quad T_{q, k}((F * G)_q^{(g_1, g_2; h_1, h_2)})(y) = (T_{q, k_1}(F) * T_{q, k_2}(G))_q^{(g_1, g_2; h_3, h_4)}(y).$$

In views of the assumptions in Theorems 5.3 and 5.5, we must check that there exist solutions $\{g_1, g_2, k, k_1, k_2, h_1, h_2, h_3, h_4\}$ of the system

$$(6.2) \quad \begin{cases} \text{(i)} & g_1 g_2 k^2 + h_1 h_2 = 0 \text{ in } L_2[0, T], \\ \text{(ii)} & m_L(\text{supp}(h_3) \cap \text{supp}(h_4)) = 0, \\ \text{(iii)} & \mathbf{s}(g_1 k, h_1) = \mathbf{s}(h_3, k_1) \text{ in } L_2[0, T], \\ & \text{i.e., } g_1^2(t) k^2(t) + h_1^2(t) = h_3^2(t) + k_1^2(t) \text{ } m_L\text{-a.e. } t \in [0, T], \\ \text{(iv)} & \mathbf{s}(g_2 k, h_2) = \mathbf{s}(h_4, k_2) \text{ in } L_2[0, T], \\ & \text{i.e., } g_2^2(t) k^2(t) + h_2^2(t) = h_4^2(t) + k_2^2(t) \text{ } m_L\text{-a.e. } t \in [0, T]. \end{cases}$$

to establish equation (6.1) above.

Throughout the remainder of this paper, we give examples of the solution sets of the system (6.2).

Example 6.1. *The set $\{g_1, g_2, k, k_1, k_2, h_1, h_2, h_3, h_4\}$ of functions in $L_2[0, T]$ with*

$$\begin{aligned} g_1(t) &= 2 \cos\left(\frac{2\pi t}{T}\right) \chi_{[0, T/2]}(t), & g_2(t) &= \left[3 - 4 \sin^2\left(\frac{2\pi t}{T}\right)\right] \chi_{[T/2, T]}(t), \\ k(t) &= \sin\left(\frac{2\pi t}{T}\right), & k_1(t) &= \sin\left(\frac{4\pi t}{T}\right), & k_2(t) &= \sin\left(\frac{6\pi t}{T}\right), \\ h_1(t) &= \chi_{[T/2, T]}(t), & h_2(t) &= \chi_{[0, T/2]}(t), \\ h_3(t) &= \cos\left(\frac{4\pi t}{T}\right) \chi_{[T/2, T]}(t), & h_4(t) &= \cos\left(\frac{6\pi t}{T}\right) \chi_{[0, T/2]}(t), \end{aligned}$$

is a solution of the system (6.2). The functions above are defined m_L -a.e. on $[0, T]$.

Example 6.2. *Given positive integers l, m and n with $l < m < n$, let*

$$\begin{aligned} g_1(t) &= \sin\left(\frac{l\pi t}{T}\right), & g_2(t) &= \sin\left(\frac{m\pi t}{T}\right), & k(t) &= \cos\left(\frac{n\pi t}{T}\right), \\ k_1(t) &= \sqrt{2} \sin\left(\frac{l\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right) \chi_B(t), & k_2(t) &= \sqrt{2} \sin\left(\frac{m\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right) \chi_B(t), \\ h_1(t) &= \sin\left(\frac{l\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right), & h_2(t) &= -\sin\left(\frac{m\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right), \\ h_3(t) &= \sqrt{2} \sin\left(\frac{l\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right) \chi_A(t), & h_4(t) &= \sqrt{2} \sin\left(\frac{m\pi t}{T}\right) \cos\left(\frac{n\pi t}{T}\right) \chi_A(t). \end{aligned}$$

Then the set $\mathbf{S} = \{g_1, g_2, k, k_1, k_2, h_1, h_2, h_3, h_4\}$ is a solution of the system (6.2).

In fact, the solution set \mathbf{S} can be obtained by the following procedures. First, let $\{A, B\}$ be a measurable partition of $[0, T]$ with $m_L(A) > 0$ and $m_L(B) > 0$. Next, given any functions g_1, g_2 and k in $BV[0, T]$, let

$$\begin{aligned} h_1(t) &= g_1(t)k(t), & h_2(t) &= -g_2(t)k(t), \\ h_3(t) &= \sqrt{2}g_1(t)k(t)\chi_A(t), & h_4(t) &= \sqrt{2}g_2(t)k(t)\chi_A(t), \\ k_1(t) &= \sqrt{2}g_1(t)k(t)\chi_B(t), & k_2(t) &= \sqrt{2}g_2(t)k(t)\chi_B(t). \end{aligned}$$

Then one can see that the set $\{g_1, g_2, k, h_1, h_2, h_3, h_4, k_1, k_2\}$ is a solution of the system (6.2).

Example 6.3. *Let $\mathcal{H} = \{h_n\}_{n=1}^\infty$ be the sequence of Haar functions on $[0, T]$. It is well-known that \mathcal{H} is a complete orthonormal set on $L_2[0, T]$ which consists of nonsmooth functions.*

Consider the intervals $A = [0, T/2]$ and $B = [T/2, T]$. Then, for each $n \in \mathbb{N}$ with $n \neq 1$, either $\text{supp}(h_n) \subset [0, T/2]$ or $\text{supp}(h_n) \subset [T/2, T]$.

Let

$$P_1 = \{n \in \mathbb{N} : \text{supp}(h_n) \subset [0, T/2]\}$$

and

$$P_2 = \{n \in \mathbb{N} : \text{supp}(h_n) \subset [T/2, T]\}.$$

Then, clearly,

$$\bigcup_{n \in P_1} \text{supp}(h_n) = [0, T/2] \quad \text{and} \quad \bigcup_{n \in P_2} \text{supp}(h_n) = [T/2, T].$$

Also, let

$$\mathcal{H}^A = \{\chi_A\} \cup \{h_n/\sqrt{2} : n \in P_1\} \equiv \{h_n^A\}_{n=1}^\infty$$

and

$$\mathcal{H}^B = \{\chi_B\} \cup \{h_n/\sqrt{2} : n \in P_2\} \equiv \{h_n^B\}_{n=1}^\infty.$$

Then it follows that

- (i) \mathcal{H}^A is a complete orthonormal set in $L_2(A) = L_2[0, T/2]$; and
- (ii) \mathcal{H}^B is a complete orthonormal set in $L_2(B) = L_2[T/2, T]$.

As discussed in Example 6.2 above, given g_1, g_2 and k in $BV[0, T]$, let

$$\begin{aligned} h_1(t) &= g_1(t)k(t) & a.e. \quad t \in [0, T], \\ h_2(t) &= -g_2(t)k(t) & a.e. \quad t \in [0, T]. \end{aligned}$$

In these settings, for $j \in \{1, 2\}$, let

$$\sum_{n=1}^\infty \alpha_n^{(j)} h_n^A \equiv \sum_n \alpha_n^{(j)} h_n^A$$

be the Fourier series of $\sqrt{2}g_j k$ with respect to \mathcal{H}^A on $[0, T/2]$, and let

$$\sum_{n=1}^\infty \beta_n^{(j)} h_n^B \equiv \sum_n \beta_n^{(j)} h_n^B$$

be the Fourier series of $\sqrt{2}g_j k$ with respect to \mathcal{H}^B on $[T/2, T]$. Then, one can see that

- (i) $g_1 g_2(t) k^2(t) + h_1(t) h_2(t) = g_1 g_2(t) k^2(t) - g_1 g_2(t) k^2(t) = 0$,
- (ii) $m_L(\text{supp}(h_3) \cap \text{supp}(h_4)) = m_L(A \cap B) = 0$,
- (iii)

$$\begin{aligned} g_1^2(t) k^2(t) + h_1^2(t) &= 2g_1^2(t) k^2(t) = [\sqrt{2}g_1(t)k(t)]^2 \\ &= [\sqrt{2}g_1(t)k(t)\chi_A(t) + \sqrt{2}g_1(t)k(t)\chi_B(t)]^2 \\ &= 2g_1^2(t)k^2(t)\chi_A(t) + 2g_1^2(t)k^2(t)\chi_B(t) \\ &= \left(\sum_{n=1}^\infty \alpha_n^{(1)} h_n^A \right)^2(t) + \left(\sum_{n=1}^\infty \beta_n^{(1)} h_n^B \right)^2(t), \end{aligned}$$

(iv)

$$\begin{aligned}
g_2^2(t)k^2(t) + h_2^2(t) &= 2g_2^2(t)k^2(t) = [\sqrt{2}g_2(t)k(t)]^2 \\
&= [\sqrt{2}g_2(t)k(t)\chi_A(t) + \sqrt{2}g_1(t)k(t)\chi_B(t)]^2 \\
&= 2g_2^2(t)k^2(t)\chi_A(t) + 2g_2^2(t)k^2(t)\chi_B(t) \\
&= \left(\sum_{n=1}^{\infty} \alpha_n^{(2)} h_n^A \right)^2(t) + \left(\sum_{n=1}^{\infty} \beta_n^{(2)} h_n^B \right)^2(t)
\end{aligned}$$

for a.e. $t \in [0, T]$. Thus, given functions g_1 , g_2 and k in $BV[0, T]$, it follows that

$$\begin{aligned}
&T_{q,k}((F * G)_\lambda^{(g_1, g_2; g_1 k, -g_2 k)})(y) \\
&= (T_{q, \sum_n \beta_n^{(1)} h_n^B}(F) * T_{q, \sum_n \alpha_n^{(2)} h_n^B}(G))_q^{(g_1, g_2; \sum_n \alpha_n^{(1)} h_n^A, \sum_n \beta_n^{(2)} h_n^A)}(y)
\end{aligned}$$

for s-a.e. $y \in C_0[0, T]$.

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